CONVERGENCE TO TYPE I DISTRIBUTION OF THE EXTREMES OF SEQUENCES DEFINED BY RANDOM DIFFERENCE EQUATION

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Abstract. We study the extremes of a sequence of random variables (R_n) defined by the recurrence $R_n = M_n R_{n-1} + q$, $n \ge 1$, where R_0 is arbitrary, (M_n) are iid copies of a non–degenerate random variable M, $0 \le M \le 1$, and $q > 0$ is a constant. We show that under mild and natural conditions on M the suitably normalized extremes of (R_n) converge in distribution to a double exponential random variable. This partially complements a result of de Haan, Resnick, Rootzén, and de Vries who considered extremes of the sequence (R_n) under the assumption that $\mathbb{P}(M > 1) > 0$.

1. Introduction

We consider a special case of the following random difference equation

$$R_n = Q_n + M_n R_{n-1}, \quad n \ge 1 \tag{1.1}$$

where R_0 is arbitrary and (Q_n, M_n), $n \ge 1$, are i.i.d. copies of a two dimensional random vector (Q, M), and (Q_n, M_n) is independent of R_{n-1}. Later on we specialize our discussion to a non–degenerate M, and $Q \equiv q$, a positive constant. Much of the impetus for studying equations like (1.1) stems from numerous applications of such schemes in mathematics and other disciplines of science. We refer to [7, 20] for examples of fields in which equation (1.1) have been of interest. Further examples of more recent applications are mentioned in [12], and for examples of statistical issues arising in studying solutions of (1.1) see [2].

A fundamental theoretical result that goes back to Kesten [14] asserts that if

$$E \ln |M| < 0 \quad \text{and} \quad E \ln |Q| < \infty \tag{1.2}$$

then the sequence (R_n) converges in distribution to a random variable R, which necessarily satisfies distributional identity

$$R \stackrel{d}{=} MR + Q \tag{1.3}$$

(see also [20] for a detailed discussion of the convergence properties of (R_n)). In the same paper Kesten showed that if $P(|M| > 1) > 0$ and (1.2) holds then, under some mild

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additional conditions on M and Q the limiting distribution is always heavy–tailed, that is, $\mathbb{P}(|R| > t) \sim Ct^{-\kappa}$ for a suitably chosen $\kappa > 0$. A different proof of this results was given by Goldie in [9]. By contrast, it was shown in [10] that in the complementary case $|M| \le 1$ if $|Q| \le q$ then the tail of R has no slower than exponential decay.

Interestingly, much more work has been done in the heavy–tailed situation. This is perhaps at least partially caused by the fact that many of the processes appearing in applications (for example GARCH processes in financial mathematics) are in fact heavy–tailed. Nonetheless, the case $|M| \le 1$ and $Q \equiv q$ contains a number of interesting situations, including the class of Vervaat perpetuities, see e.g. [20]. Vervaat perpetuities correspond to M being a Beta$(\alpha, 1)$ random variable for some $\alpha > 0$ and $Q = 1$ in which case one gets

$$R \stackrel{d}{=} 1 + M_1 + M_1M_2 + M_1M_2M_3 + \dots \tag{1.4}$$

(some authors prefer not to have a 1 at the beginning which corresponds to taking $Q = M$). Particular cases of Vervaat perpetuities include the Dickman distribution appearing in number theory (see [6]), in the analysis of the limiting distribution of `Quickselect` algorithm (see [16]), and in the limit theory of functionals of success epochs in iid sequences of random variables [19, Section 4.7]. Further connections are referenced in [13] and we refer there for more information. For recent work on perfect simulation of Vervaat perpetuities see [8] or [5].

In this note we will be interested in the extremal behavior of the sequence (R_n). For any sequence of random variables (Y_n) we let (Y_n^*) be the sequence of partial maxima, i.e. $Y_n^* = \max_{k\le n} Y_k$, $n \ge 1$. With this notation, we will seek constants (a_n) and (b_n) such that for all x

$$P(a_n(R_n^* - b_n) \le x) \longrightarrow G(x), \quad n \to \infty, \tag{1.5}$$

where G is a non–degenerate distribution function.

Under the assumptions that $\mathbb{P}(M > 1) > 0$ the extremes of the sequence (R_n) when both M and Q are non–negative were studied in [4] and were shown to converge (after suitable normalization) to Fréchet (i.e. Type II) distribution with parameter κ. Here, we consider the complementary case, namely of a light-tailed limiting distribution R. Of course, in this situation one expects convergence in (1.5) to a Gumbel (i.e. a double–exponential or Type I) distribution, provided that there is convergence at all. The latter need not be the case, however. Indeed, if $Q = 1$ and M has a two–point distribution $\mathbb{P}(M = 1) = p = 1 - \mathbb{P}(M = 0)$ then as is seen from (1.4) R has a geometric distribution with parameter $1-p$ and thus no constants (a_n), (b_n) exist for which (1.5) holds for a non–degenerate distribution G (see [15, Example 1.7.15]). Our main aim here is to show that under fairly general and natural conditions on M (and for a degenerate Q) (1.5) does hold for suitable constants (a_n), (b_n) and a double exponential distribution $G(x) = \exp(-e^{-x})$, $-\infty < x < \infty$.

2. Extremal behavior

Following the authors of [4] we assume that both M and Q are non–negative. As we mentioned earlier we assume that $Q = q > 0$ is a constant. So, we consider

$$R_n = M_n R_{n-1} + q, \quad n \geq 1, \quad R_0 - \text{given}, \tag{2.6}$$

where M_n and R_{n-1} on the right–hand side are independent and where (M_n) is a sequence of iid copies of a random variable M satisfying

$$0 \leq M \leq 1, \quad M - \text{non-degenerate}. \tag{2.7}$$

(The non–degeneracy assumption is to eliminate the possibility that R itself is degenerate.) Clearly, this is more than (1.2) and thus implies the convergence in distribution of (R_n). Furthermore, it has been known since [10] that in that case the tail of the limiting variable R is no heavier than exponential. Note that if M is bounded away from 1 then R is actually a bounded random variable. To exclude this situation we assume that the right endpoint of M is 1, that is that

$$\sup\{x : \mathbb{P}(M > x) > 0\} = 1. \tag{2.8}$$

Finally, we need to eliminate the possibility that R is a geometric variable. To this end it is enough to assume that

$$\mathbb{P}(M = 0) = 0, \tag{2.9}$$

since this guarantees that the distribution of R is continuous (see e.g. [1, Theorem 1.3]).

We will prove the following theorem

Theorem 1. *Let (R_n) satisfy (2.6) with M satisfying (2.7)–(2.9). Then there exist sequences (a_n) (b_n) such that for every real x*

$$\mathbb{P}(a_n(R_n^* - b_n) \leq x) \to \exp(-e^{-x}), \quad \textit{as } n \to \infty.$$

3. proof of Theorem 1

We first outline our proof which generally follows the approach of [4] (see also references therein for earlier developments). Writing out (2.6) explicitly we see that

$$R_n = q + qM_n + qM_nM_{n-1} + \cdots + qM_n \ldots M_2 + M_n \ldots M_1 R_0. \tag{3.10}$$

Under our assumption (2.7) (as a matter of fact, under the first part of (1.2) as well) the product $\prod_{k=1}^n M_k$ goes to 0 a.s. Consequently, the extremal behavior of (R_n) is the same regardless of the choice of the initial variable R_0. It is particularly convenient to choose R_0 so that it satisfies (1.3) as then so does every R_k, $k \geq 1$, making the sequence (R_n) stationary. Extremal behavior of stationary sequences is quite well understood (see e.g. [15, Chapter 3]) and we will take advantage of that. To find the extremal behavior of (R_n) one has to do three things:

(i) analyse the extremal behavior of the associated independent sequence $(\hat{R}_n)$ consisting of iid random variables equidistributed with R,
(ii) verify that the sequence (R_n) satisfies the $D(u_n)$ condition for sequences (u_n) of the form $u_n = b_n + x/a_n$, for any x and suitably chosen sequences (a_n), (b_n), and
(iii) show that the sequence (R_n) has the extremal index and find its value.

Some of the difficulties with carrying out this program are caused by the fact that, contrary to the heavy – tailed situation, much less is known about the tail asymptotics in the case of light tails. A notable exception are Vervaat perpetuities (see [19, Section] for a discussion). General results on light–tail case are scarce (see [10, 12, 11]) and less precise than Kesten's result in the heavy – tailed situation. As a consequence, less precise information about the norming constants (a_n) (b_n) will be available. Our substitute for Kesten's result will be two–sided bounds obtained recently in [11].

We will treat the three items above in separate subsections.

3.1. **Associated independent sequence.** We appeal to the general theory of extremes as described in e.g. [15, Chapter 1]. First, we know from [1, Theorem 1.3] that (2.9) and non–degeneracy assumption on M imply that R has continuous distribution function F_R. Therefore, the condition (1.7.3) of Theorem 1.7.13 of [15] is satisfied and thus, for every $x > 0$ there exist $u_n = u_n(x)$ such that

$$\lim_{n\to\infty} n\mathbb{P}(R > u_n) = e^{-x}. \tag{3.11}$$

In fact, since R is continuous u_n may be taken to be

$$u_n(x) = F_R^{-1}(1 - \frac{e^{-x}}{n}),$$

where F_R is the probability distribution function of R. The question now is whether u_n's may be chosen to be linear functions of x i.e. whether there exist constants a_n and b_n, $n \ge 1$ such that for $x > 0$ we have

$$u_n(x) = \frac{x}{a_n} + b_n, \qquad n \ge 1. \tag{3.12}$$

To address that question we will utilize a recent result of [11] which states that there exist absolute constants c_i, $i = 0, 1, 2, 3$ such that for sufficiently large $y > 0$

$$\exp\{c_0 y \ln p_{\frac{c_1}{y}}\} \le \mathbb{P}(R > y) \le \exp\{c_2 y \ln p_{\frac{c_3}{y}}\},$$

where, following [10], for $0 < \delta < 1$ we set

$$p_\delta = \mathbb{P}(1-\delta < M \le 1) = 1 - F_M(1-\delta) \quad \text{and} \quad p_0 = \lim_{\delta\to 0} p_\delta = \mathbb{P}(M = 1). \tag{3.13}$$

Notice that by (2.8) p_δ is strictly positive for $\delta \in (0, 1)$. Now, if

$$\mathbb{P}(R > u_n) = \frac{e^{-x}}{n},$$

then

$$\exp\{c_0 u_n \ln p_{\frac{c_1}{u_n}}\} \le \frac{e^{-x}}{n}.$$

Therefore, if w_n are chosen so that

$$\exp\{c_0 w_n \ln p_{\frac{c_1}{w_n}}\} = \frac{e^{-x}}{n},$$

then $u_n \ge w_n$. By the same argument, if v_n are such that

$$\exp\{c_2 v_n \ln p_{\frac{c_3}{v_n}}\} = \frac{e^{-x}}{n},$$

then $\mathbb{P}(R > v_n) \le \frac{e^{-x}}{n}$ so that $u_n \le v_n$. Hence for every $x > 0$

$$w_n(x) \le u_n(x) \le v_n(x)$$

and thus for every $n \ge 1$ there would exist $0 \le \alpha_n \le 1$ such that

$$u_n = \alpha_n w_n + (1-\alpha_n) v_n.$$

If both (v_n) and (w_n) were linear, say,

$$w_n(x) = \frac{x}{a'_n} + b'_n, \quad v_n(x) = \frac{x}{a''_n} + b''_n,$$

for some (a'_n), (b'_n), (a''_n), and (b''_n) then (3.12) would hold with

$$a_n = \left(\frac{\alpha_n}{a'_n} + \frac{1-\alpha_n}{a''_n}\right)^{-1} \quad \text{and} \quad b_n = \alpha_n b'_n + (1-\alpha_n) b''_n.$$

It therefore suffices to show the existence of linear norming for partial maxima of iid random variables (W_n) whose common distribution F_W satisfies

$$1 - F_W(y) = \exp\{c_0 y \ln p_{c_1/y}\}, \quad \text{for} \quad y \ge y_0,$$

where $p_{c_1/y}$ is given by (3.13) for some fixed random variable M satisfying (2.7)–(2.9).

In accordance with [15, Theorem 1.5.1] to show that

$$\mathbb{P}(a'_n(W_n - b'_n) \le x) \to \exp(-e^{-x}),$$

holds for every real x, the constants (a'_n) and $b'_n)$ must be constructed so that for every such x

$$n(1 - F_W(b'_n + x/a'_n)) \to e^{-x}, \quad \text{as} \quad n \to \infty,$$

i.e. that

$$n \exp\left\{c_0 (b'_n + \frac{x}{a'_n}) \ln p_{\frac{c_1}{b'_n + x/a'_n}}\right\} \to e^{-x}, \quad \text{as} \quad n \to \infty. \tag{3.14}$$

Choose b'_n so that

$$c_0 b'_n \ln p_{c_1/b'_n} = -\ln n. \tag{3.15}$$

Then the left–hand side of (3.14) is

$$\exp\left\{c_0 b_n'\left(\ln p_{\frac{c_1}{b_n'+x/a_n'}}-\ln p_{c_1/b_n'}\right)\left(1+\frac{x}{a_n'b_n'}\right)+c_0\frac{x}{a_n'}\ln p_{c_1/b_n'}\right\}.$$

To choose (a_n') first note that the difference of logarithms in the first summand is negative. Hence, if for any n, $a_n' \le -K\ln p_{c_1/b_n'}$ for some $K < c_0$ then the exponent is no more than $-xc_0/K < -x$. Therefore, for any admissible choice of (a_n') we must have $\liminf_n a_n'/\ln p_{c_1/b_n'} \le -c_0$ which implies in particular that $a_n'b_n' \to \infty$. Thus, the exponent in the above formula is asymptotic to

$$c_0 b_n'(\ln p_{\frac{c_1}{b_n'+x/a_n'}}-\ln p_{c_1/b_n'})+c_0\frac{x}{a_n'}\ln p_{c_1/b_n'}.$$

We can further assume that for each n $1-c/b_n'$ is a differentiability point of F_M and that the derivative, f_M, is finite at $1-c_1/b_n'$. It then follows that the exponent is asymptotic to

$$-c_0\frac{c_1 x}{a_n'b_n'p_{c_1/b_n'}}f_M(1-\frac{c_1}{b_n'})+c_0\frac{x}{a_n'}\ln p_{c_1/b_n'}$$

and thus we may choose

$$a_n' = c_0\left(\frac{c_1}{b_n'p_{c_1/b_n'}}f_M(1-\frac{c_1}{b_n'})-\ln p_{c_1/b_n'}\right). \tag{3.16}$$

3.2. $D(u_n)$ **condition.** To check that $D(u_n)$ condition holds for sequences of the form $b_n + x/a_n$ we proceed in the same fashion as [4, proof of Theorem 2.1]; the argument there was, in turn, based on [17, proof of Lemma 3.1]. Recall that, according to [15, Lemma 3.2.1(ii)] it suffices to show that if $1 \le i_1 < \cdots < i_r < j_1 < \cdots < j_s \le n$ are such that $j_1 - i_r \ge \lambda n$ for $\lambda > 0$ then

$$\mathbb{P}(\bigcap_{k=1}^r\{R_{i_k}\le u_n\}\cap\bigcap_{m=1}^s\{R_{j_m}\le u_n\})-\mathbb{P}(\bigcap_{k=1}^r\{R_{i_k}\le u_n\})\mathbb{P}(\bigcap_{k=1}^r\{R_{i_k}\le u_n\})\to 0,$$

as $n\to\infty$. Set $I=\{i_1,\dots,i_r\}$ and $J=\{j_1,\dots,j_s\}$ and for any set A of positive integers let $R_A^*=\max_{a\in A}R_a$.

It follows from (2.6) that for $j > i$ we have

$$\begin{aligned}R_j &= q+qM_j+\cdots+qM_j\dots M_{i+2}+M_j\dots M_{i+1}R_i\\ &=: S_{j,i}+M_j\cdot\dots\cdot M_{i+1}R_i,\end{aligned}$$

where, for $j > i$ we have set

$$S_{j,i}:=q+qM_j+\cdots+qM_j\dots M_{i+2}.$$

Hence, for any $\epsilon_n > 0$ we obtain

$$\begin{aligned}
\{R_J^* \le u_n\} &= \bigcap_{j\in J}\{S_{j,i_r} + M_j \dots M_{i_r+1}R_{i_r} \le u_n\} \\
&\supset \bigcap_{j\in J}\{S_{j,i_r} \le u_n - \epsilon_n\} \cap \{M_j \dots M_{i_r+1}R_{i_r} \le \epsilon_n\} \\
&= \bigcap_{j\in J}\{S_{j,i_r} \le u_n - \epsilon_n\} \setminus \bigcup_{j\in J}\{M_j \dots M_{i_r+1}R_{i_r} > \epsilon_n\}.
\end{aligned}$$

Note that R_k and $S_{n,m}$ are independent whenever, $m \ge k$ so that $\{R_i :\ i \in I\}$ and $\{S_{j,i_r} :\ j \in J\}$ are independent, and hence we get

$$P(R_I^* \le u_n, R_J^* \le u_n) \ge P(R_I^* \le u_n)P(S_{J,i_r}^* \le u_n - \epsilon_n) - P(\bigcup_{j\in J} M_j \dots M_{i_r+1}R_{i_r} > \epsilon_n).$$

Also,

$$\{S_{J,i_r}^* \le u_n - \epsilon_n\} \supset \{R_J^* \le u_n - 2\epsilon_n\} \cap \bigcap_{j\in J}\{M_j \dots M_{i_r+1}R_{i_r} \le \epsilon_n\},$$

which further leads to

$$P(R_I^* \le u_n, R_J^* \le u_n) \ge P(R_I^* \le u_n)P(R_J^* \le u_n - 2\epsilon_n) - 2P(\bigcup_{j\in J} M_j \dots M_{i_r+1}R_{i_r} > \epsilon_n).$$

By essentially the same argument we also get

$$P(R_I^* \le u_n, R_J^* \le u_n) \le P(R_I^* \le u_n)P(R_J^* \le u_n + 2\epsilon_n) + 2P(\bigcup_{j\in J} M_j \dots M_{i_r+1}R_{i_r} > \epsilon_n).$$

Combining these two estimates we obtain

$$\begin{aligned}
&|P(R_I^* \le u_n, R_J^* \le u_n) - P(R_I^* \le u_n)P(R_J^* \le u_n)| \\
&\quad \le \max\{P(R_J^* \le u_n) - P(R_J^* \le u_n - 2\epsilon_n), P(R_J^* \le u_n + 2\epsilon_n) - P(R_J^* \le u_n)\} \\
&\qquad + 2P(\bigcup_{j\in J} M_j \dots M_{i_r+1}R_{i_r} > \epsilon_n).
\end{aligned}$$

Thus condition $D(u_n)$ will be verified once we show that both terms in the sum on the right–hand side vanish as $n \to \infty$. To handle the first term we use stationarity and the fact that $j_s \le n$ to find that the maximum above is bounded by

$$\sum_{j\in J} P(u_n - 2\epsilon_n \le R_j \le u_n + 2\epsilon_n) \le nP(u_n - 2\epsilon_n \le R \le u_n + 2\epsilon_n).$$

Recall that (u_n) satisfy (3.11) and (3.12). Thus, setting $\epsilon_n = \epsilon/a_n$ with $\epsilon > 0$ sufficiently small we get

$$nP(u_n - 2\epsilon_n \le R \le u_n + 2\epsilon_n) \to e^{-(x-2\epsilon)} - e^{-(x+2\epsilon)} = O(\epsilon).$$

Turning attention to the second term, using $M_k \le 1$ we see that

$$\begin{aligned}\mathbb{P}(\bigcup_{j\in J} M_j \dots M_{i_r+1}R_{i_r} > \epsilon/a_n) &\le \sum_{j\in J}\mathbb{P}(M_j \dots M_{i_r+1}R_{i_r} > \epsilon/a_n)\\ &\le nP(M_{j_1-i_r}\dots M_1R_0 > \epsilon/a_n).\end{aligned}$$

Intersect the event underneath this probability with $\{R > 2b_n\}$ and its complement to see that this term is bounded by

$$n\mathbb{P}(R > 2b_n) + n\mathbb{P}(M_{j_1-i_r}\dots M_1 > \epsilon/(2a_nb_n)). \tag{3.17}$$

Furthermore, since for any $T > 0$ and sufficiently large n, $2b_n = b_n + \frac{a_nb_n}{a_n} > b_n + T/a_n$, the first term (3.17) is bounded by

$$n\mathbb{P}(R > 2b_n) \le n\mathbb{P}(R > b_n + T/a_n) \to e^{-T},$$

and thus goes to 0 upon letting $T \to \infty$. Turning to the second term in (3.17) we see that by Markov's inequality and independence of M_k's it is bounded by

$$\frac{2na_nb_n}{\epsilon}(EM)^{j_1-i_r}. \tag{3.18}$$

We need to see that this vanishes as $n \to \infty$. Recall that $EM < 1$ and $j_1 - i_r \ge \lambda n$ where $\lambda > 0$, so that $(\mathbb{E}M)^{j_1-i_r}$ decays exponentially fast in n. Furthermore,

$$a_nb_n \le K\max\{a'_n, a''_n\}\cdot\max\{b'_n, b''_n\}$$

Recall that b'_n and b''_n satisfy (3.15) (with different constants). Thus they both are $O(\ln n)$ as are $\ln p_{c_1/b'_n}$ and $\ln p_{c_3/b''_n}$. Hence,

$$a'_n \le K\left(\frac{c_1}{b'_n}f_M(1-\frac{c_1}{b'_n})\frac{1}{p_{c_1/b'_n}} + \ln n\right).$$

Since f_M is an integrable function, we may assume that $\frac{c_1}{b'_n}f_M(1-c_1/b'_n) = O(1)$ as $n \to \infty$. Finally, recall that (b'_n) satisfies (3.15). Therefore,

$$p_{c_1/b'_n} = \exp(-\frac{\ln n}{c_0b'_n}) = n^{-1/c_0b'_n} \ge n^{-\alpha}, \quad \alpha > 0,$$

where the last inequality follows from the fact that $b'_n \to \infty$ as $n \to \infty$ which is evident from (3.15). It follows that na_nb_n has a polynomial growth in n and thus that for every $\epsilon > 0$ (3.18) goes to 0 as $n \to \infty$.

3.3. **Extremal index.** We establish the following fact about the extremal index of (R_n). It implies, in particular, that if M does not have an atom at 1, then the extremal behavior of (R_n) is exactly the same as it would be for independent R_n's.

Proposition 2. *Let (R_n) be a stationary sequence satisfying the recurrence* (2.6)*. Then (R_n) has the extremal index θ whose value is*

$$\theta = \limsup_n \mathbb{P}(MR + q \le u_n | R > u_n) = 1 - p_0 = 1 - \mathbb{P}(M = 1).$$

Proof. Again following the authors of [4] we rely on Theorem 4.1 of Rootzén [18]. Since we have shown that $D(u_n)$ holds for every sequence u_n of the form $b_n + x/a_n$, $x > 0$, it remains to verify condition (4.3) of that theorem i.e. to show that

$$\limsup_{n\to\infty} |\mathbb{P}(R_{\lceil n\epsilon\rceil} \le u_n | R_0 > u_n) - \theta| \to 0, \quad \text{as} \quad \epsilon \searrow 0.$$

To this end, for given $\epsilon > 0$, let $m := m_\epsilon := \lceil n\epsilon\rceil$. Then

$$\mathbb{P}(R_m^* \le u_n | R_0 > u_n) = \mathbb{P}(R_m \le u_n | R_{m-1}^* \le u_n, R_0 > u_n)\mathbb{P}(R_{m-1}^* \le u_n | R_0 > u_n).$$

By Markov property, for $m \ge 2$ the first probability on the right–hand side is

$$\mathbb{P}(R_m \le u_n | R_{m-1} \le u_n) = \mathbb{P}(M_m R_{m-1} + q \le u_n | R_{m-1} \le u_n) = \mathbb{P}(MR + q \le u_n | R \le u_n).$$

Continuing in the same fashion we find that

$$\begin{aligned} \mathbb{P}(R_m^* \le u_n | R_0 > u_n) &= \mathbb{P}^{m-1}(MR + q \le u_n | R \le u_n)\mathbb{P}(R_1 \le u_n | R_0 > u_n) \\ &= (1 - \mathbb{P}(MR + q > u_n | R \le u_n))^{m-1}\, \mathbb{P}(MR + q \le u_n | R > u_n). \end{aligned}$$

So, clearly

$$\limsup_n \mathbb{P}(R_m^* \le u_n | R_0 > u_n) \le \limsup_n \mathbb{P}(MR + q \le u_n | R > u_n).$$

On the other hand,

$$\begin{aligned} n\mathbb{P}(MR + q > u_n | R \le u_n) &= n\frac{\mathbb{P}(MR + q > u_n, R \le u_n)}{\mathbb{P}(R \le u_n)} \le n\frac{\mathbb{P}(MR + q > u_n)}{1 - \mathbb{P}(R > u_n)} \\ &= n\frac{\mathbb{P}(R > u_n)}{1 - \mathbb{P}(R > u_n)} \to e^{-x}, \end{aligned}$$

as $n \to \infty$ by the very choice of (u_n). Thus

$$\limsup_n n\mathbb{P}(MR + q > u_n | R \le u_n) \le e^{-x} =: c < \infty$$

so that

$$\liminf_n\, (1 - \mathbb{P}(MR + q > u_n | R \le u_n))^{m-1} \ge e^{-c\epsilon}$$

and hence

$$\limsup_n \mathbb{P}(R_m^* \le u_n | R_0 \le u_n) \ge e^{-c\epsilon} \limsup_n \mathbb{P}(MR + q \le u_n | R > u_n).$$

It follows that

$$\begin{aligned} &\lim_{\epsilon\searrow 0} \limsup_{n\to\infty} \left\{(1 - \mathbb{P}(MR + q > u_n | R \le u_n))^{m-1}\, \mathbb{P}(MR + q \le u_n | R > u_n)\right\} \\ &\qquad = \limsup_{n\to\infty} \mathbb{P}(MR + q \le u_n | R > u_n). \end{aligned}$$

We now turn to evaluating

$$\limsup_{n\to\infty} \mathbb{P}(MR + q \le u_n | R > u_n).$$

It is clear that if $p_0 > 0$ then for every n such that $u_n \ge q$ we have $\mathbb{P}(MR+q \le u_n | R > u_n) = 1 - p_0$ so assume that $p_0 = 0$ and write

$$\mathbb{P}(MR+q \le u_n | R > u_n) = 1 - \mathbb{P}(MR+q > u_n | R > u_n) = 1 - \frac{\mathbb{P}(MR+q > u_n,\ R > u_n)}{\mathbb{P}(R > u_n)}.$$

It remains to show that the numerator in the last expression is of lower order than the denominator. To do that, let (t_n) be a sequence converging to infinity but in such a way that $t_n = o(b_n)$. Then

$$\begin{aligned}\mathbb{P}(MR+q > u_n, R > u_n) &= \int_{u_n}^{\infty} \mathbb{P}(Mt+q > u_n) dF_R(t) \\ &= \left(\int_{u_n}^{u_n+t_n} + \int_{u_n+t_n}^{\infty}\right) \mathbb{P}(Mt+q > u_n) dF_R(t).\end{aligned}$$

Note that the probability underneath the integral is an increasing function of t. Bounding it trivially by 1 in the second term we see that this term is bounded by $\mathbb{P}(R > u_n + t_n)$. This can be further bounded by

$$\mathbb{P}(R > b_n + \frac{x}{a_n} + t_n) = \mathbb{P}(R > b_n + \frac{x + a_n t_n}{a_n}) \le \mathbb{P}(R > b_n + \frac{x+T}{a_n}),$$

whenever $a_n t_n \ge T$. It follows by the choice of (u_n) and $D(u_n)$ condition that

$$\frac{\mathbb{P}(R > u_n + t_n)}{\mathbb{P}(R > u_n)} \le e^{-T},$$

for arbitrarily large T and sufficiently large n and thus it vanishes as $n \to \infty$. The first integral is bounded by

$$\mathbb{P}(M(u_n + t_n) + q > u_n)\mathbb{P}(u_n < R < u_n + t_n) \le \mathbb{P}(M > 1 - \frac{t_n + q}{u_n + t_n})\mathbb{P}(R > u_n).$$

Since the first term goes to $p_0 = 0$ as $n \to \infty$, we see that this term is $o(\mathbb{P}(R > u_n))$ as $n \to \infty$. This shows that the extremal index is 1 when $p_0 = 0$ and completes the proof. □

4. Remarks

1. The main drawback of Theorem 1 is that it does not give a good handle on the norming constants (a_n) and (b_n). This is generally caused by a lack of precise information about the tails of the limiting random variable R. However, even in the rare cases in which more precise information about tails of R is available, the formulas seem to be too complicated to make the precise statements about (a_n) and (b_n) practical. For example, when $q = 1$ and M has Beta$(\alpha, 1)$ distribution, $\alpha > 0$, (i.e. R is a Vervaat perpetuity) then Vervaat [19, Theorem 4.7.7] (based on earlier arguments of de Bruijn [3]) found the expression for the density of R. This, in principle, could be used to get precise enough asymptotics of the tail function of R and thus determine the asymptotic values of (b_n) and (a_n). However,

the nature of these formulas, makes obtaining explicit asymptotic expressions for (a_n) and (b_n) difficult if not impossible. As far as we know, Vervaat perpetuities provide the only class of examples (within our restrictions on M and Q) for which the asymptotics of the tail function is known. On the other hand, Theorem 1 typically gives the order of the magnitude of (a_n) and (b_n).

2. The expression (3.16) for (a'_n) often simplifies to $a'_n \sim -c_0 \ln p_{c_1/b'_n}$ (with corresponding simplification for (a_n)). This will happen, for example, whenever $p_0 = 0$ and $\delta f_M(1-\delta)/p_\delta$ is bounded as $\delta \to 0$, in particular, when M is Beta(α, β) random variable, $\alpha, \beta > 0$. In that case, b'_n may be chosen to be asymptotic to $\frac{\ln n}{c_0 \beta \ln\ln n}$ and then $a'_n \sim c_0 \beta \ln\ln n$. Hence, (a_n) and (b_n) are of order $\ln\ln n$ and $\ln n/\ln\ln n$, respectively. Note that Vervaat perpetuity corresponds to $\beta = 1$ and Dickman distribution to $\alpha = \beta = 1$.

3. There are, however, situations for which the above remark is not true. The following situation was considered in [12, Theorem 6]. Let M have density given by

$$f_M(t) = K \exp\{-\frac{1}{(1-t^r)^{1/(r-1)}}\}, \quad 1 < r < \infty, \quad 0 < t < 1,$$

where $K = K_r$ is a normalizing constant. Then, as $\delta \searrow 0$,

$$p_\delta \sim (1-(1-\delta)^r)^{r/(r-1)} \exp\{-(1-(1-\delta)^r)^{-1/(r-1)}\} \sim (r\delta)^{r/(r-1)} \exp\{-(r\delta)^{-1/(r-1)}\},$$

so that

$$\frac{c_1 f_M(1-c_1/b'_n)}{b'_n p_{c_1/b'_n}} \sim \left(\frac{b'_n}{c_1 r^r}\right)^{1/(r-1)}.$$

On the other hand,

$$-\ln p_{c_1/b'_n} \sim \left(\frac{b'_n}{c_1}\right)^{1/(r-1)} + \frac{r}{r-1}\ln(b'_n/rc_1) = \left(\frac{b'_n}{c_1}\right)^{1/(r-1)} \left(1 + O(\frac{\ln b'_n}{b_n'^{1/(r-1)}})\right),$$

so that both terms appearing in (3.16) are of the same order. Here again, the norming constants (a_n), (b_n) in Theorem 1 may be determined up to absolute multiplicative factors and are of order $(\ln n)^{1/r}$ and $(\ln n)^{(r-1)/r}$, respectively.

4. Consider another example from [12] in which

$$f_M(t) = K \exp\left(-\int_{1-t}^{1} \frac{e^{1/s}}{s} ds\right), \quad 0 < t < 1.$$

Then (see [12, Lemma 8]) $\ln p_\delta \sim -\delta e^{1/\delta}$ as $\delta \to 0$. Similarly, one can check that

$$\frac{\delta f_M(1-\delta)}{p_\delta} \sim \frac{\delta e^{-\delta e^{1/\delta}} e^{\delta e^{1/\delta}}}{\delta e^{-1/\delta}} = e^{1/\delta},$$

so this time the first term in the expression (3.16) is of higher order than the second. It follows from the asymptotics above that $a'_n \sim (\ln n)/c_0 c_1$ and $b'_n \sim c_1 \ln\ln n$ and hence (a_n), (b_n) are of order $\ln n$ and $\ln\ln n$, respectively.

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